11. Let V $= \{0\}$ consist of a single vector 0 and define $0 + 0 = 0$ and $c0 = 0$ for each scalar c in F. Prove that V is a vector space over F. (V is called the **zero vector space.**)

12. A real-valued function f defined on the real line is called an **even function** if $f(-t) = f(t)$ for each real number t. Prove that the set of even functions defined on the real line with the operations of addition and scalar multiplication defined in Example 3 is a vector space.

13. Let V denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of V and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2) \quad \text{and} \quad c(a_1, a_2) = (ca_1, a_2).$$

Is V a vector space over R with these operations? Justify your answer.

14. Let V $= \{(a_1, a_2, \ldots, a_n) \colon a_i \in C \text{ for } i = 1, 2, \ldots n\}$; so V is a vector space over C by Example 1. Is V a vector space over the field of real numbers with the operations of coordinatewise addition and multiplication?

15. Let V $= \{(a_1, a_2, \ldots, a_n) \colon a_i \in R \text{ for } i = 1, 2, \ldots n\}$; so V is a vector space over R by Example 1. Is V a vector space over the field of complex numbers with the operations of coordinatewise addition and multiplication?

16. Let V denote the set of all $m \times n$ matrices with real entries; so V is a vector space over R by Example 2. Let F be the field of rational numbers. Is V a vector space over F with the usual definitions of matrix addition and scalar multiplication?

17. Let V $= \{(a_1, a_2) \colon a_1, a_2 \in F\}$, where F is a field. Define addition of elements of V coordinatewise, and for $c \in F$ and $(a_1, a_2) \in$ V, define

$$c(a_1, a_2) = (a_1, 0).$$

Is V a vector space over F with these operations? Justify your answer.

18. Let V $= \{(a_1, a_2) \colon a_1, a_2 \in R\}$. For $(a_1, a_2), (b_1, b_2) \in$ V and $c \in R$, define

$$\ldots 3b_2) \quad \text{and} \quad c(a_1, a_2) = (ca_1, ca_2).$$

19. Let $V = \{(a_1, a_2) : a_1, a_2 \in R\}$. Define addition of elements of V coordinatewise, and for (a_1, a_2) in V and $c \in R$, define

$$c(a_1, a_2) = \begin{cases} (0,0) & \text{if } c = 0 \\ \left(ca_1, \dfrac{a_2}{c}\right) & \text{if } c \neq 0. \end{cases}$$

Is V a vector space over R with these operations? Justify your answer.

20. Let V be the set of sequences $\{a_n\}$ of real numbers. (See Example 5 for the definition of a sequence.) For $\{a_n\}, \{b_n\} \in V$ and any real number t, define

$$\{a_n\} + \{b_n\} = \{a_n + b_n\} \quad \text{and} \quad t\{a_n\} = \{ta_n\}.$$

Prove that, with these operations, V is a vector space over R.

21. Let V and W be vector spaces over a field F. Let

$$Z = \{(v, w) : v \in V \text{ and } w \in W\}.$$

Prove that Z is a vector space over F with the operations

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \quad \text{and} \quad c(v_1, w_1) = (cv_1, cw_1).$$

22. How many matrices are there in the vector space $M_{m \times n}(Z_2)$? (See Appendix C.)

1.3 SUBSPACES

In the study of any algebraic structure, it is of interest to examine subsets that possess the same structure as the set under consideration. The appropriate notion of substructure for vector spaces is introduced in this section.

Definition. A subset W of a vector space V over a field F is called a **subspace** of V if W is a vector space over F with the operations of addition and scalar multiplication defined on V.

10. Prove that $W_1 = \{(a_1, a_2, \ldots, a_n) \in F^n : a_1 + a_2 + \cdots + a_n = 0\}$ is a subspace of F^n, but $W_2 = \{(a_1, a_2, \ldots, a_n) \in F^n : a_1 + a_2 + \cdots + a_n = 1\}$ is not.

11. Is the set $W = \{f(x) \in P(F) : f(x) = 0 \text{ or } f(x) \text{ has degree } n\}$ a subspace of $P(F)$ if $n \geq 1$? Justify your answer.

12. An $m \times n$ matrix A is called **upper triangular** if all entries lying below the diagonal entries are zero, that is, if $A_{ij} = 0$ whenever $i > j$. Prove that the upper triangular matrices form a subspace of $M_{m \times n}(F)$.

13. Let S be a nonempty set and F a field. Prove that for any $s_0 \in S$, $\{f \in \mathcal{F}(S, F) : f(s_0) = 0\}$, is a subspace of $\mathcal{F}(S, F)$.

14. Let S be a nonempty set and F a field. Let $\mathcal{C}(S, F)$ denote the set of all functions $f \in \mathcal{F}(S, F)$ such that $f(s) = 0$ for all but a finite number of elements of S. Prove that $\mathcal{C}(S, F)$ is a subspace of $\mathcal{F}(S, F)$.

15. Is the set of all differentiable real-valued functions defined on R a subspace of $C(R)$? Justify your answer.

16. Let $C^n(R)$ denote the set of all real-valued functions defined on the real line that have a continuous nth derivative. Prove that $C^n(R)$ is a subspace of $\mathcal{F}(R, R)$.

17. Prove that a subset W of a vector space V is a subspace of V if and only if $W \neq \varnothing$, and, whenever $a \in F$ and $x, y \in W$, then $ax \in W$ and $x + y \in W$.

18. Prove that a subset W of a vector space V is a subspace of V if and only if $0 \in W$ and $ax + y \in W$ whenever $a \in F$ and $x, y \in W$.

19. Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

20.† Prove that if W is a subspace of a vector space V and $w_1, w_2, \ldots, w_n$ are in W, then $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in W$ for any scalars $a_1, a_2, \ldots, a_n$.

21. Show that the set of convergent sequences $\{a_n\}$ (i.e., those for which $\lim_{n\to\infty} a_n$ exists) is a subspace of the vector space V in Exercise 20 of Section 1.2.

22. Let F_1 and F_2 be fields. A function $g \in \mathcal{F}(F_1, F_2)$ is called an **even function** if $g(-t) = g(t)$ for each $t \in F_1$ and is called an **odd function** if $g(-t) = -g(t)$ for each $t \in F_1$. Prove that the set of all even functions in $\mathcal{F}(F_1, F_2)$ and the set of all odd functions in $\mathcal{F}(F_1, F_2)$ are subspaces of $\mathcal{F}(F_1, F_2)$.

†A dagger means that this exercise is essential for a later section.

The following definitions are used in Exercises 23–30.

Definition. *If S_1 and S_2 are nonempty subsets of a vector space* V*, then the* **sum** *of S_1 and S_2, denoted $S_1 + S_2$, is the set $\{x+y\colon x \in S_1 \text{ and } y \in S_2\}$.*

Definition. *A vector space* V *is called the* **direct sum** *of* W_1 *and* W_2 *if* W_1 *and* W_2 *are subspaces of* V *such that* $\mathsf{W}_1 \cap \mathsf{W}_2 = \{0\}$ *and* $\mathsf{W}_1 + \mathsf{W}_2 = \mathsf{V}$. *We denote that* V *is the direct sum of* W_1 *and* W_2 *by writing* $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$.

23. Let W_1 and W_2 be subspaces of a vector space V.

(a) Prove that $\mathsf{W}_1 + \mathsf{W}_2$ is a subspace of V that contains both W_1 and W_2.

(b) Prove that any subspace of V that contains both W_1 and W_2 must also contain $\mathsf{W}_1 + \mathsf{W}_2$.

24. Show that F^n is the direct sum of the subspaces

$$\mathsf{W}_1 = \{(a_1, a_2, \ldots, a_n) \in \mathsf{F}^n\colon a_n = 0\}$$

and

$$\mathsf{W}_2 = \{(a_1, a_2, \ldots, a_n) \in \mathsf{F}^n\colon a_1 = a_2 = \cdots = a_{n-1} = 0\}.$$

25. Let W_1 denote the set of all polynomials $f(x)$ in $\mathsf{P}(F)$ such that in the representation

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

we have $a_i = 0$ whenever i is even. Likewise let W_2 denote the set of all polynomials $g(x)$ in $\mathsf{P}(F)$ such that in the representation

$$g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0,$$

we have $b_i = 0$ whenever i is odd. Prove that $\mathsf{P}(F) = \mathsf{W}_1 \oplus \mathsf{W}_2$.

26. In $\mathsf{M}_{m\times n}(F)$ define $\mathsf{W}_1 = \{A \in \mathsf{M}_{m\times n}(F)\colon A_{ij} = 0 \text{ whenever } i > j\}$ and $\mathsf{W}_2 = \{A \in \mathsf{M}_{m\times n}(F)\colon A_{ij} = 0 \text{ whenever } i \leq j\}$. ($\mathsf{W}_1$ is the set of all upper triangular matrices defined in Exercise 12.) Show that $\mathsf{M}_{m\times n}(F) = \mathsf{W}_1 \oplus \mathsf{W}_2$.

27. Let V denote the vector space consisting of all upper triangular $n \times n$ matrices (as defined in Exercise 12), and let W_1 denote the subspace of ... $\mathsf{W}_1 \oplus \mathsf{W}_2$, where

2. Solve the following systems of linear equations by the method introduced in this section.

(a)
$$\begin{aligned} 2x_1 - 2x_2 - 3x_3 \qquad\quad &= -2 \\ 3x_1 - 3x_2 - 2x_3 + 5x_4 &= 7 \\ x_1 - x_2 - 2x_3 - x_4 &= -3 \end{aligned}$$

(b)
$$\begin{aligned} 3x_1 - 7x_2 + 4x_3 &= 10 \\ x_1 - 2x_2 + x_3 &= 3 \\ 2x_1 - x_2 - 2x_3 &= 6 \end{aligned}$$

(c)
$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 5 \\ x_1 + 4x_2 - 3x_3 - 3x_4 &= 6 \\ 2x_1 + 3x_2 - x_3 + 4x_4 &= 8 \end{aligned}$$

(d)
$$\begin{aligned} x_1 + 2x_2 + 2x_3 \qquad\quad &= 2 \\ x_1 \qquad\quad + 8x_3 + 5x_4 &= -6 \\ x_1 + x_2 + 5x_3 + 5x_4 &= 3 \end{aligned}$$

(e)
$$\begin{aligned} x_1 + 2x_2 - 4x_3 - x_4 + x_5 &= 7 \\ -x_1 \qquad\quad + 10x_3 - 3x_4 - 4x_5 &= -16 \\ 2x_1 + 5x_2 - 5x_3 - 4x_4 - x_5 &= 2 \\ 4x_1 + 11x_2 - 7x_3 - 10x_4 - 2x_5 &= 7 \end{aligned}$$

(f)
$$\begin{aligned} x_1 + 2x_2 + 6x_3 &= -1 \\ 2x_1 + x_2 + x_3 &= 8 \\ 3x_1 + x_2 - x_3 &= 15 \\ x_1 + 3x_2 + 10x_3 &= -5 \end{aligned}$$

3. For each of the following lists of vectors in R^3, determine whether the first vector can be expressed as a linear combination of the other two.

(a) $(-2, 0, 3), (1, 3, 0), (2, 4, -1)$
(b) $(1, 2, -3), (-3, 2, 1), (2, -1, -1)$
(c) $(3, 4, 1), (1, -2, 1), (-2, -1, 1)$
(d) $(2, -1, 0), (1, 2, -3), (1, -3, 2)$
(e) $(5, 1, -5), (1, -2, -3), (-2, 3, -4)$
(f) $(-2, 2, 2), (1, 2, -1), (-3, -3, 3)$

determine whether the first poly-

5. In each part, determine whether the given vector is in the span of S.

(a) $(2,-1,1)$, $S=\{(1,0,2),(-1,1,1)\}$
(b) $(-1,2,1)$, $S=\{(1,0,2),(-1,1,1)\}$
(c) $(-1,1,1,2)$, $S=\{(1,0,1,-1),(0,1,1,1)\}$
(d) $(2,-1,1,-3)$, $S=\{(1,0,1,-1),(0,1,1,1)\}$
(e) $-x^3+2x^2+3x+3$, $S=\{x^3+x^2+x+1,x^2+x+1,x+1\}$
(f) $2x^3-x^2+x+3$, $S=\{x^3+x^2+x+1,x^2+x+1,x+1\}$
(g) $\begin{pmatrix}1 & 2\\ -3 & 4\end{pmatrix}$, $S=\left\{\begin{pmatrix}1 & 0\\ -1 & 0\end{pmatrix},\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix},\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}\right\}$
(h) $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $S=\left\{\begin{pmatrix}1 & 0\\ -1 & 0\end{pmatrix},\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix},\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}\right\}$

6. Show that the vectors $(1,1,0)$, $(1,0,1)$, and $(0,1,1)$ generate F^3.

7. In F^n, let e_j denote the vector whose jth coordinate is 1 and whose other coordinates are 0. Prove that $\{e_1,e_2,\ldots,e_n\}$ generates F^n.

8. Show that $\mathsf{P}_n(F)$ is generated by $\{1,x,\ldots,x^n\}$.

9. Show that the matrices

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\quad \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\quad \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},\quad \text{and}\quad \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$

generate $\mathsf{M}_{2\times 2}(F)$.

10. Show that if

$$M_1=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\quad M_2=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},\quad \text{and}\quad M_3=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

then the span of $\{M_1,M_2,M_3\}$ is the set of all symmetric 2×2 matrices.

11.† Prove that $\text{span}(\{x\})=\{ax\colon a\in F\}$ for any vector x in a vector space. Interpret this result geometrically in R^3.

12. Show that a subset W of a vector space V is a subspace of V if and only if $\text{span}(\mathsf{W})=\mathsf{W}$.

... are subsets of a vector space V such that $S_1 \subseteq S_2$, ... $\text{span}(S_1)=\mathsf{V}$,